TEAR-OFF VERSUS GLOBAL EXISTENCE FOR A STRUCTURED MODEL OF ADHESION MEDIATED BY TRANSIENT ELASTIC LINKAGES.

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Abstract. We consider a microscopic model for friction mediated by transient elastic linkages introduced in [9, 10]. In the present study we prove existence and uniqueness of a solution to the coupled system under weaker hypotheses. The theory we present covers the case where the off-rate of linkages is unbounded but increasing at most linearly with respect to the mechanical load.

The time of existence is typically bounded and corresponds to tear-off where the moving binding site does not have any bonds with the substrate. However, under additional assumptions on the external force we prove global in time existence of a solution that consequently stays attached to the substrate.

Contents

1. Introduction

Adhesion forces at the cellular and intra-cellular scales play an important role in several phenomenons as cell motility (see [12] and references therein), or cancer growth [14]. In [12, 11] the authors derive a complete model for a moving network of actin filaments polymerizing near the boundary of the cell and depolymerizing close to the nucleus, providing biologically plausible steady-state configurations of the cell shape. The main advantage of this method is that the parameters used are experimentally easy to obtain if not already available in the literature [6, 3, 4, 7, 13]. The adhesion and the stretching between filaments are written as friction terms obtained through a formal limits of a delayed system of equations. Indeed, let ε be a dimensionless parameter denoting the ratio of the typical lifetime of bonds vs. the overall timescale of the model. The asymptotic limit is obtained assuming that the rate of linkage turnover becomes great as well as the stiffness of the bonds (typically as $O(1/\varepsilon)$). The rigorous justification of the limit $\varepsilon \to 0$ is the ultimate goal of our investigations [9, 10]. Nevertheless the highly non-linear nature of the delayed model leads to consider already the case of a fixed value of ε. In this article we show that, even then, the data of the problem determins the well-posedness of the model : the balance between the on-rate of the linkages and the force exerted on the adhesion point is essential. Mathematically this is seen since, according to this balance, either we can show blow up in finite time or global existence. Physically this means that pulling too strong

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the adhesion site causes a tear-off, and that our model is able to reproduce this feature. Experimentally this is observed and used in order to measure the probability distribution of unbinding forces [1, 2, 16, 5].

More precisely, this study is concerned with a system of equations which describe the evolution of the time-dependent position of a single binding site as it moves on a 1D-substrate. External forces f act on the moving point-object while it is attached to the substrate through continuously remodeling elastic linkages which represent the effect of transiently attaching protein bonds. Their age distribution is denoted by $\rho_\varepsilon = \rho_\varepsilon(t,a)$ where $a \geq 0$ denotes the age of linkages and $t \geq 0$ denotes time. Here we treat ε as a fixed constant, which we keep in our notation to maintain consistency with previous studies, in which we were analyzing the convergence with respect to ε [9, 10].

In [12, 11] the following structured model for the turnover of protein bonds with age distribution $\rho_\varepsilon = \rho_\varepsilon(t,a)$ for age $a \geq 0$ has been established,

$$\begin{cases} \varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta\, \rho_\varepsilon = 0\,, & t > 0\,,\ a > 0\,, \\ \rho_\varepsilon(t, a=0) = \beta(t)\,(1-\mu_0)\,, & t > 0\,, \\ \rho_\varepsilon(t=0,a) = \rho_{I,\varepsilon}(a)\,, & a \geq 0\,, \end{cases} \tag{1.1}$$

where $\mu_0(t) := \int_0^\infty \rho_\varepsilon(t,\tilde{a})\, d\tilde{a}$ and the on-rate of bonds is a given coefficient β times a factor, that takes into account saturation of the moving binding site with linkages. This system is coupled to the elongation $u_\varepsilon = u_\varepsilon(t,a)$ of the linkage through

$$\zeta := \xi(|u_\varepsilon(t,a)|)\,, \tag{1.2}$$

according to which the off-rate $\xi(u) > 0$ is a real, positive function of the elongation of the linkage u_ε. In [10] we introduced the following age-structured model for the evolution of the elongation u_ε,

$$\begin{cases} \varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = \dfrac{1}{\mu_0}\left(\varepsilon \partial_t f + \displaystyle\int_0^\infty \left(\zeta(u_\varepsilon) u_\varepsilon \rho_\varepsilon\right)(t,\tilde{a})\, d\tilde{a} \right)\,, & t > 0\,,\ a > 0\,, \\ u_\varepsilon(t, a=0) = 0\,, & t > 0\,, \\ u_\varepsilon(t=0,a) = u_{I,\varepsilon}(a)\,, & a \geq 0\,, \end{cases} \tag{1.3}$$

where $f = f(t) \in \mathbb{R}$ is the external force acting on the binding site. In [10] we had shown that the system (1.3) is equivalent to an integral equation for the position of the binding site $z_\varepsilon = z_\varepsilon(t)$ itself [9], namely

$$\begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_0^\infty \left(z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)\right) \rho_\varepsilon(t,a)\, da = f(t)\,, & t \geq 0\,, \\ z_\varepsilon(t) = z_p(t)\,, & t < 0\,, \end{cases} \tag{1.4}$$

where the known past positions are given by the Lipschitz function $z_p(t) \in \mathbb{R}$ for $t < 0$. The correspondance between (1.4) and (1.3) is made through

$$u_\varepsilon = \frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon}\,.$$

Note that the equation (1.4) has been the original result of the mathematical modeling since it represents the balance of external and elatic forces acting on the binding site. On the other hand, in [10] it has turned out to be beneficial to work on the system (1.1), (1.3) instead, since it allowed to derive powerful a-priori estimates on u_ε.

The analysis in the older studies [9] and [10] relied on the existence of an upper bound $\zeta_{\max}$ of the function ζ. In [10], for fixed ε, we prove existence and uniqueness of weak solutions of the coupled system (1.1), (1.2), (1.3). In the simpler semi-coupled case when $\zeta(t,a)$ is a given bounded function we give as well a convergence result as $\varepsilon \to 0$. The analysis of (1.1) in [9] relied on the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_0[\rho - \rho_\infty] \leq -\zeta_{\min}/\varepsilon \mathcal{H}_0[\rho-\rho_\infty]$ where

$$\mathcal{H}_0[\rho](t) := \int_{\mathbb{R}_+} |\rho(t,a)| da + \left| \int_{\mathbb{R}_+} \rho(t,a) da \right|, \tag{1.5}$$

and where ρ_∞ is the stationary solution of (1.1). For the analysis of (1.3) in [10] we established the a-priori estimate (Lemma 4.1, [10])

$$\int_{\mathbb{R}_+} \rho_\varepsilon(t,a) |u_\varepsilon(t,a)| da \leq \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a) |u_{I,\varepsilon}(a)| da + \int_0^t |\partial_t f(\tilde{t})| d\tilde{t}\,. \tag{1.6}$$

Note that both results, decay of the functional (1.5) and the *a-priori* estimate (1.6), do not rely on the existence of an upper bound of ζ and therefore do hold in the framework of this paper.

It is the aim of the present study to relax the hypothesis of boundedness of ζ. This represents a major improvement of the analysis, because the lower bound of the total mass $\mu_0(t)$ strongly depends on $\zeta_{\max}$ and the analytical arguments in [10] do rely heavily on this control. Furthermore the upper bound $\zeta_{\max}$ had major importance in the fixed point argument used in [10] to prove the global existence result since we used it to control the non-linear right hand side in (1.3).

In addition to deepening the analysis, unboundedness of the off-rate is the natural scenario from the application point of view. A typical situation is Bell's law, i.e. an exponential increase of the off-rate as the elastic linker is extended, $\zeta = \zeta_0 \exp(|u_\varepsilon|)$ (cf. [15, 8]). However, this strongly non-linear scenario is still out of reach of the rigorous mathematical analysis that we present in this study which relies on ζ being a (globally) Lipschitz continous function as it's main technical assumption.

The right hand side of (1.3) for a given function u,

$$g_u(t) := \frac{1}{\mu_{0,u}} \left\{ \varepsilon \partial_t f + \int_{\mathbb{R}_+} \zeta(u(t,a))\, \varrho_u(t,a)\, u(t,a)\, da \right\} ,$$

where ϱ_u solves (1.1) with $\zeta = \zeta(u)$ and $\mu_{0,u} := \int_{\mathbb{R}_+} \varrho_u(t,a)\, da$, can become infinite if either $\mu_{0,u}$ vanishes or $\int_{\mathbb{R}_+} \zeta(u)\, u\, \varrho_u\, da$ blows up. We define the modified right hand side

$$\overline{\overline{g}}_u := \frac{1}{\max(\mu_{0,u}, \underline{\mu})} \left\{ \varepsilon \partial_t f + \max\left(-\overline{p}, \min\left(\overline{p}, \int_{\mathbb{R}_+} \zeta(u)\, \varrho_u\, u\, da \right)\right) \right\} ,$$

where $\underline{\mu}$ and $\overline{p}$ are two strictly positive arbitrary constants. The strategy to prove our existence result is first to establish existence and uniqueness of a solution of this modified problem using a fixed point argument in the space

$$X_T := \left\{ u \in L^\infty_{\mathrm{loc}}((0,T) \times \mathbb{R}_+) \text{ s.t. } \sup_{t \in (0,T)} \|u(t,a)\, \omega(a)\|_{L^\infty_a} < \infty \right\} \tag{1.7}$$

defined for any specific time $T > 0$, where the weight function is

$$\omega(a) := \frac{1}{1+a} . \tag{1.8}$$

To this end we introduce the map $\Phi : v \in X_T \mapsto u \in X_T$ where, given v, we solve (1.1) with $\zeta = \zeta(v)$ and obtain the age distribution ρ_v. Then we look for the solution of the problem :

$$\begin{cases} \varepsilon \partial_t u + \partial_a u = \overline{\overline{g}}_v(t) , \quad t > 0 , \quad a > 0 , \\ u(t,0) = 0 , \qquad\qquad\qquad\qquad t > 0 , \\ u(0,a) = u_{I,\varepsilon}(a) , \qquad\qquad\quad a \geq 0 , \end{cases} \tag{1.9}$$

to obtain $u \in X_T$. The right hand side of (1.9) becomes a bounded function whose bounds depend on the cut-offs $\underline{\mu}$ and $\overline{p}$. This allows to prove contraction of the map Φ on a time interval that is sufficiently small. Due to the uniform bounds this process can be iterated to obtain (ϱ_w, w) a unique, global in time, solution. Then we establish a uniform bound on $p(t) := \int_{\mathbb{R}_+} \zeta(w) w \rho_w da$, the second integral term in g_w. This shows that for $\overline{p}$ sufficiently large with respect to $1/\underline{\mu}$, $p(t)$ never reaches $\overline{p}$ so that the solution (ρ_w, w) satisfies also a simple-cut-of problem where $\overline{\overline{g}}_u$ can be replaced by $\overline{g}_u$ defined as :

$$\overline{g}_u := \frac{1}{\max(\mu_{0,u}, \underline{\mu})} \left\{ \varepsilon \partial_t f + \int_{\mathbb{R}_+} \zeta(u)\, \varrho_u\, u\, da \right\} .$$

In a second step, we prove that if additional assumptions hold, this solution never reaches the cut-off value $\underline{\mu}$. Otherwise, we give a lower bound to the time span during which the cut-off is not reached. In both cases the solution of the modified problem is also the unique solution to the original system (1.1)-(1.3) either globally in time or on the finite interval of time.

More precisely, in Section 4, we analyze the dependence of the lower bound of $\mu_{0,u}$ with respect to the $L^\infty(0,T)$ norm of $\overline{g}_u$. This naturally leads to local existence results for the original problem (1.1)-(1.3) in Section 5 by providing a minimal time for which the solution (ρ_w, w) does not reach the cut-off value $\underline{\mu}$.

Even stronger results are rigorously obtained in Sections 6 and 7 generalising a straightforward computation in the special case where $\zeta(u) = 1 + |u|$ and assuming that u_ε remains strictly positive. In this case, integrating (1.1) in age, and using the fact that (1.4) transforms in $\int_{\mathbb{R}_+} \rho_\varepsilon(t,a) u_\varepsilon(t,a) da = f(t)$, we obtain that

$$\varepsilon \partial_t \mu_0 - \beta(1-\mu_0) + \mu_0 + f = 0 ,$$

which can be solved directly. This provides immediately the bounds

$$\min\left(\mu_0(0), \frac{\beta_{\min} - f_{\max}}{\beta_{\max}+1}\right) \leq \mu_0(t) \leq \mu_0(0)\left(1 - \frac{t}{t_0}\right),$$

where

$$t_0 := \frac{\varepsilon}{\beta_{\min}+1} \ln\left(1 + \frac{\mu_0(0)(\beta_{\min}+1)}{f_{\min} - \beta_{\max}}\right)$$

and leads to a strictly positive lower bound of μ_0 when $\beta_{\min} > f_{\max}$, whereas if $f_{\min} > \beta_{\max}$, the time t_0 is well defined and the binding site tears off, i.e. $\mu_0(t)$ becomes zero, at $t = t_0$. These basic ideas provide global existence results (Section 6) vs. tear-off results (Section 7) under more general assumptions on ζ.

2. Technical assumptions, preliminary results and a-priori estimates

2.1. Hypotheses.

Assumption 2.1. *a) There exists a minimal value* $\zeta_{\min}$ *s.t.* $\zeta(u) \geq \zeta_{\min} > 0$, $\forall u \in \mathbb{R}$.
b) The derivative of ζ *is bounded* i.e. $|\zeta'(u)| \leq \zeta_{\mathrm{Lip}}$, $\forall u \in \mathbb{R}$.
c) The function f *is Lipschitz continuous on* $[0,T]$ *for any positive fixed* T.

Remark 2.2. *Note that with this definition we do not allow more than a linear growth for* ζ. *But in contrast to* [9, 10] *one has no hypothesis concerning boundedness on* ζ.

As in [10] we assume also some hypotheses on the initial and boundary data of (1.1):

Assumption 2.3. *The initial condition* $\rho_{I,\varepsilon} \in L^\infty_a(\mathbb{R}_+)$ *is*

(i) nonnegative, i.e. $\rho_{I,\varepsilon}(a) \geq 0$, *a.e. in* $\mathbb{R}_+$.
(ii) Moreover, the total initial population satisfies

$$0 < \int_0^\infty \rho_{I,\varepsilon}(a) da < 1 ,$$

(iii) and higher moments are bounded,

$$0 < \int_0^\infty a^p \rho_{I,\varepsilon}(a)\, da \leq c_p , \quad \text{for } p = 1,2 ,$$

where c_p *are positive constants depending only on* p.

Assumption 2.4. *For* β *we assume that*
a) $\beta = \beta(t)$ *is a continuous function,*
b) $0 < \beta_{\min} \leq \beta(t) \leq \beta_{\max}$ *for all positive times* t.

We detail hereafter those results from [9] that are still valid in the weaker frame of Assumptions 2.1, 2.3 and 2.4.

Theorem 2.5. *We suppose that* u *is a given function in* X_T. *Let Assumptions 2.1, 2.3 and 2.4 hold, then for every fixed* ε *there exists a unique solution* $\varrho \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}^2_+)$ *of the problem* (1.1)*, with the off-rate* $\zeta(t,a) := \zeta(u(t,a))$. *It satisfies* (1.1) *in the sense of characteristics, namely*

$$\varrho(t,a) = \begin{cases} \beta(t-\varepsilon a)\left(1 - \int_0^\infty \varrho(\tilde a, t-\varepsilon a)\, d\tilde a\right) \times \\ \qquad \times \exp\left(-\int_0^a \zeta(\tilde a, t - \varepsilon(a-\tilde a))\, d\tilde a\right) , & \text{when } a < t/\varepsilon , \\ \rho_{I,\varepsilon}(a - t/\varepsilon) \exp\left(-\frac{1}{\varepsilon}\int_0^t \zeta((\tilde t - t)/\varepsilon + a, \tilde t)\, d\tilde t\right) , & \text{if } a \geq t/\varepsilon . \end{cases} \tag{2.1}$$

Lemma 2.6. *Under the same assumptions as in Theorem 2.5, let ϱ be the unique solution of problem (1.1), then it satisfies a weak formulation of this problem, namely*

$$\int_0^\infty \int_0^T \varrho(t,a)\left(\varepsilon\partial_t\varphi + \partial_a\varphi - \zeta\varphi\right)\, dt\, da - \varepsilon \int_0^\infty \varrho(t,a)\varphi(t=T,a)\, da + \\ + \int_0^T \varrho(t,a=0)\,\varphi(t,0)\, dt + \varepsilon \int_0^\infty \rho_{I,\varepsilon}(a)\varphi(t=0,a)\, da = 0 \,, \tag{2.2}$$

for every $T > 0$ and every test function $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}_+)$.

Following the same argumentation as Lemma 2.2 in [9] one has

Lemma 2.7. *Under the same assumptions as in Theorem 2.5, it holds that $\mu_0(t) < 1$ for any time. This in turn implies that $\varrho(t,a) \geq 0$ for almost every (t,a) in $\mathbb{R}_+^2$.*

For $p \in \mathbb{N}$ we define the p-th moment of the solution ρ_ε of (1.1)

$$\mu_p(t) := \int_0^\infty a^p \varrho(t,a)\, da \,.$$

Then, following the same argumentation as Lemma 2.2 in [9], one has

Lemma 2.8. *Under the same assumptions as in Theorem 2.5,*

$$\mu_p(t) \leq \mu_{p,\max} \quad \textit{for} \quad p = 1,2 \,,$$

where the generic constants $\mu_{p,\max}$ read :

$$\mu_{p,\max} := \sum_{\ell=0}^{p} \frac{p!}{\ell!\,\zeta_{\min}^{p-\ell}}\mu_\ell(0) + \frac{p!}{\zeta_{\min}^p}\frac{\beta_{\max}}{\beta_{\min}+\zeta_{\min}}.$$

Proof. When $p = 0$ we simply integrate (1.1) with respect to age :

$$\varepsilon\partial_t\mu_0 + \beta\mu_0 + \int_{\mathbb{R}_+} \zeta\rho_\varepsilon da = \beta$$

as ζ is bounded from below and using Gronwall's lemma one has :

$$\mu_0(t) \leq \mu_0(0) + \frac{\beta_{\max}}{\beta_{\min}+\zeta_{\min}}$$

For any integer p we then write :

$$\varepsilon\partial_t\mu_p + \zeta_{\min}\mu_p - p\mu_{p-1} \leq 0$$

which using Gronwall's lemma gives

$$\|\mu_p\|_{L^\infty(0,T)} \leq \mu_p(0) + \frac{p}{\zeta_{\min}}\|\mu_{p-1}\|_{L^\infty(0,T)}.$$

By recursion, one proves the claim. □

Proposition 2.9. *Under Assumptions 2.1, 2.3 and 2.4, setting $\hat\rho := \varrho_2 - \varrho_1$ where ϱ_2 and ϱ_1 solve (1.1) with off-rates $\zeta(w_2)$ (resp. $\zeta(w_1)$) where w_2 (resp. w_1) is a function in X_T, we find that*

$$\mathcal{H}_0[\hat\rho](t) \leq c_0(1-\exp(\zeta_{\min}t/\varepsilon))\|\hat w\|_{X_t} \,, \quad \forall t \in (0,T) \,,$$

where $\hat w := w_2 - w_1$, $c_0 := \frac{2}{\zeta_{\min}}\zeta_{\mathrm{Lip}}\mu_{1,\max}$, $\mu_{1,\max}$ being the bound on the first moment of ϱ_1.

Proof. The proof follows the same lines as for Lemma 3.2 and Lemma 3.3 in [9] based on the system satisfied by $\hat\rho$,

$$\begin{cases} \varepsilon\partial_t\hat\rho + \partial_a\hat\rho + \zeta_2\hat\rho = -\hat\zeta\varrho_1 & t>0, a>0, \\ \hat\rho(t,0) = -\beta(t)\displaystyle\int_{\mathbb{R}_+}\hat\rho(t,\tilde a)\, d\tilde a, & t>0, \\ \hat\rho(0,a) = 0, & a>0, \end{cases}$$

where $\hat\zeta := \zeta(w_2) - \zeta(w_1)$. □

For $k \geq 1$ we define

$$\mathcal{H}_k[\rho] := \int_{\mathbb{R}_+} (1+a)^k \rho(t,a) da$$

for these functionals one has :

Proposition 2.10. *Under the same hypotheses as in the previous proposition, and if moreover*

$$\int_{\mathbb{R}_+} (1+a)^\ell \rho_{I,\varepsilon}(a)\, da < \infty, \quad \forall \ell \in \{0, k+1\},$$

then

$$\mathcal{H}_k[\hat\rho](t) \leq h_k (1 - \exp(-\zeta_{\min} t/\varepsilon)) \|\hat w\|_{X_t}, \quad \forall t \in (0,T),$$

where the constants h_k depend only on $\zeta_{\min}, \zeta_{\mathrm{Lip}}$ and on the constants $(\mu_{\ell,\max})_{\ell\in\{0,k+1\}}$ related to the bound on the ℓ-th moment of ϱ_2.

Proof. We apply a recursion argument. The case $k = 0$ is proved by Proposition 2.9. We suppose that the claim is true for $\ell \leq k-1$. We have formally that

$$\varepsilon\partial_t (1+a)^k |\hat\rho| + \partial_a (1+a)^k |\hat\rho| - k(1+a)^{k-1}|\hat\rho| + \zeta_{\min}(1+a)^k|\hat\rho| \leq |\hat\zeta|(1+a)^k \varrho_2$$

Integrating in age, one gets that

$$\varepsilon\partial_t \mathcal{H}_k[\hat\rho] - \beta|\hat\mu| + \zeta_{\min}\mathcal{H}_k[\hat\rho] \leq k\mathcal{H}_{k-1}[\hat\rho] + \zeta_{\mathrm{Lip}}\|\hat w\|_{X_t} \int_{\mathbb{R}_+} (1+a)^{k+1} \varrho_2(t,a) da$$

which is then estimated giving:

$$\varepsilon\partial_t \mathcal{H}_k[\hat\rho] + \zeta_{\min}\mathcal{H}_k[\hat\rho] \leq k\mathcal{H}_{k-1}[\hat\rho] + \zeta_{\mathrm{Lip}} C_{k+1}\|\hat w\|_{X_t} + \beta_{\max}\mathcal{H}_0[\hat\rho]$$

which using the Gronwall's Lemma gives

$$\mathcal{H}_k[\hat\rho](t) \leq \frac{1-\exp(-\zeta_{\min} t/\varepsilon)}{\zeta_{\min}} \sup_{s\in(0,t)} \left(k\mathcal{H}_{k-1}[\hat\rho](s) + \beta_{\max}\mathcal{H}_0[\hat\rho](s) + \zeta_{\mathrm{Lip}} C_{k+1} \|\hat w\|_{X_s} \right)$$

where we used, in the last estimates, the recursion hypothesis and Proposition 2.9. □

If we give ourselves $T > 0$ and a function $g \in L^\infty(0,T)$ and then we compute w as the solution in the sense of characteristics of

$$\begin{cases} \varepsilon\partial_t w + \partial_a w = g(t)\,, & t > 0\,,\ a > 0\,, \\ w(t,0) = 0\,, & t > 0\,, \\ w(0,a) = u_{I,\varepsilon}(a)\,, & a \geq 0\,. \end{cases} \tag{2.3}$$

And all along the paper we will assume that the initial condition $u_{I,\varepsilon}$ belongs to $L^\infty(\mathbb{R}_+, \omega)$. For this simple transport problem it holds that

Theorem 2.11. *If $T > 0$ and g is a function in $L^\infty(0,T)$, for any fixed ε and any $T > 0$ there exists a unique $w \in X_T$ solving problem (2.3). Moreover one has the* a priori *estimates:*

$$\|w\|_{X_T} \leq \left(\frac{T}{T+\varepsilon}\right) \|g\|_{L^\infty(0,T)} + \|u_{I,\varepsilon}\|_{L^\infty_a(\mathbb{R}_+,\omega)}$$

Moreover the maximal time of existence is infinite if $g \in L^\infty(\mathbb{R}_+)$.

3. Global existence results for cut-off problems

We solve the problem find (ϱ, w) satisfying :

$$\begin{cases} \varepsilon\partial_t \varrho + \partial_a \varrho + \zeta(w)\rho_\varepsilon = 0\,, & t > 0\,,\ a > 0\,, \\ \varrho(t,0) = \beta(t)\left(1 - \int_{\mathbb{R}_+} \varrho(t,a) da\right)\,, & t > 0\,, \\ \varrho(0,a) = \rho_{I,\varepsilon}(a)\,, & a \geq 0 \end{cases} \tag{3.1}$$

and

$$
\begin{cases}
\varepsilon \partial_t w + \partial_a w = \overline{\overline{g}}_w(t) \; , & t > 0 \; , \quad a > 0 \; , \\
w(t,0) = 0 \; , & t > 0 \; , \\
w(0,a) = u_{I,\varepsilon}(a) & a \geq 0 \; ,
\end{cases}
\tag{3.2}
$$

where we set

$$
\overline{\overline{g}}_w(t) := \frac{1}{\max(\mu_0(t), \underline{\mu})} \left(\varepsilon \partial_t f + \max\left(-\overline{p}, \min\left(\int_{\mathbb{R}_+} \zeta(w) \, \rho_\varepsilon \, w \, da \; , \; \overline{p} \right) \right) \right) ,
\tag{3.3}
$$

where $\mu_0(t) = \int_{\mathbb{R}_+} \rho_\varepsilon(t,a) \, da$. The two constants $\underline{\mu}$ and $\overline{p}$ are positive.

Lemma 3.1. *We suppose that $(\underline{\mu}, \overline{p}) \in (\mathbb{R}_+^*)^2$ and that f is Lipschitz. The function*

$$
\mathcal{G}(A,B) := \frac{1}{\max(\underline{\mu}, A)} \left\{ \varepsilon \partial_t f + \max(-\overline{p}, \min(\overline{p}, B)) \right\}
$$

is a Lipshitz function with respect to $A \in \mathbb{R}$ for any fixed $B \in \mathbb{R}$ and with respect to $B \in \mathbb{R}$ for any fixed $A \in \mathbb{R}$. The Lipschitz constants in both cases are uniform and depend only on $(\underline{\mu}, \overline{p})$.

Theorem 3.2. *We suppose that Assumptions 2.1, 2.3 and 2.4 hold. Moreover we assume that $u_{I,\varepsilon} \in L^\infty(\mathbb{R}_+, \omega)$ and $\|\partial_t f\|_{L^\infty(\mathbb{R}_+)}$ is finite and that the constants $\underline{\mu}$ and $\overline{p}$ are fixed. For any fixed time T possibly infinite, there exists a unique pair of solutions $(\varrho_w, w) \in C(0,T; L^1(\mathbb{R}_+)) \times X_T$ solving the coupled problems* (3.1)*,* (3.2) *and* (3.3)*.*

Proof. We apply the Banach fixed point Theorem to Φ mapping $w \in X_T \mapsto u \in X_T$ such that

$$
\begin{cases}
\varepsilon \partial_t u + \partial_a u = \overline{\overline{g}}_w(t), & t > 0, a > 0, \\
w(t,0) = 0, & t > 0, \\
w(0,a) = u_{I,\varepsilon}(a), & a > 0,
\end{cases}
$$

We prove that Φ is actually contractive in X_T for a time T small enough.

a) The map Φ is endomorphic. For any given $w \in X_T$ one has invariably

$$
|\overline{\overline{g}}_w| \leq \frac{1}{\underline{\mu}} \left(\varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \overline{p} \right) ,
\tag{3.4}
$$

which by the same method as in Theorem 2.11 provides a bound independent on T in X_T on u :

$$
\|u\|_{X_T} \leq \left\| \overline{\overline{g}}_w \right\|_{L^\infty(0,T)} + \|u_{I,\varepsilon}\|_{L^\infty_\omega(\mathbb{R}_+)}.
$$

b) The map Φ is a contraction. We set $\hat{g}_w := \overline{\overline{g}}_{w_2} - \overline{\overline{g}}_{w_1}$ and $\hat{\rho} := \varrho_{w_2} - \varrho_{w_1}$ and so on. Thanks to Lemma 3.1

$$
|\hat{g}_w(t)| \leq \frac{|\hat{\mu}|}{\underline{\mu}^2} \left\{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \overline{p} \right\} + \frac{1}{\underline{\mu}} \left| \left(\widehat{\int_{\mathbb{R}_+} \zeta \varrho w da} \right) \right| =: I_1 + I_2.
$$

I_1 is immediately estimated thanks to Proposition 2.9, and one has :

$$
I_1 \leq \frac{1}{\underline{\mu}^2} \left\{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \overline{p} \right\} \mathcal{H}_0[\hat{\rho}](t) \leq \frac{1}{\underline{\mu}^2} \left\{ \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \overline{p} \right\} c_0 \|\hat{w}\|_{X_t},
$$

while we decompose the difference of triple products in I_2 as :

$$
\begin{aligned}
I_2 &\leq \frac{1}{\underline{\mu}} \left| \int_{\mathbb{R}_+} \hat{\zeta} \varrho_{w_2} w_2 + \zeta_1 \hat{\rho} w_2 + \zeta_1 \varrho_{w_1} \hat{w} da \right| \\
&\leq \frac{1}{\underline{\mu}} \left(\int_{\mathbb{R}_+} \zeta_{\mathrm{Lip}} |\hat{w}| \varrho_{w_2} |w_2| da \right. \\
&\quad \left. + \left(\zeta_{\mathrm{Lip}} \|w_1\|_{X_t} + \zeta_0 \right) \left\{ \int_{\mathbb{R}_+} (1+a)^2 |\hat{\rho}| da \|w_2\|_{X_t} + \int_{\mathbb{R}_+} (1+a)^2 \varrho_{w_1} da \|\hat{w}\|_{X_t} \right\} \right) \\
&\leq c \left\{ \|\hat{w}\|_{X_t} + \mathcal{H}_2[\hat{\rho}](t) \right\} \leq \overline{c} \|\hat{w}\|_{X_t},
\end{aligned}
$$

where the constant $\overline{c}$ depends on $\zeta_{\rm Lip}, \zeta_0, (\|w_i\|_{X_t})_{i\in\{1,2\}}$, $\underline{\mu}$ and $\int_{\mathbb{R}_+} a^k \rho_{I,\varepsilon}(a)da$ for $k\in\{0,1,2\}$. Using again Theorem 2.11, one has

$$\|\hat{u}\|_{X_t} \leq \frac{t}{t+\varepsilon}\|\hat{g}_w\|_{L^\infty(0,t)} \leq \frac{t}{\varepsilon}\|\hat{g}_w\|_{L^\infty(0,t)} \leq \frac{t\overline{c}}{\varepsilon}\|\hat{w}\|_{X_t}.$$

If $T_0 < \varepsilon/\overline{c}$ then there exists a unique fixed point $w \in X_{T_0}$ of the mapping Φ.

c) Global existence for any time. We suppose that existence is established on the whole time interval $[0, T_{n-1}]$ for $n \geq 1$. We construct a fixed point for the next interval $[T_{n-1}, T_n := T_{n-1} + \Delta T_n]$ on the map $u = \Phi(v)$

$$\begin{cases} \varepsilon\partial_t u + \partial_a u = \overline{\overline{g}}_v(t), & t\in(T_{n-1},T_n), a>0,\\ u(t,0) = 0, & t\in(T_{n-1},T_n),\\ u(T_{n-1},a) = w(T_{n-1},a) & a>0. \end{cases}$$

and

$$\begin{cases} \varepsilon\partial_t \rho + \partial_a \rho + \zeta(v)\rho = 0, & t\in(T_{n-1},T_n), a>0,\\ \rho(t,0) = \beta(t)\left(1-\displaystyle\int_{\mathbb{R}_+}\rho(t,a)da\right), & t\in(T_{n-1},T_n),\\ \rho(T_{n-1},a) = \varrho(T_{n-1},a), & a>0. \end{cases}$$

If we denote the extensions to $[0, T_n]$ of (ρ, u) as :

$$\rho_e(t,a) := \begin{cases} \varrho_{\tilde{w}}(t,a) & \text{if } t\in[T_{n-1},T_n) \\ \varrho(t,a) & t\in(0,T_{n-1}] \end{cases}, \quad w_e := \begin{cases} \tilde{w}(t,a) & \text{if } t\in[T_{n-1},T_n) \\ w(t,a) & t\in(0,T_{n-1}] \end{cases},$$

where $\tilde{w} = \Phi(\tilde{w})$ and (ϱ, w) is the unique solution of (3.1)-(3.2) on $[0, T_{n-1}]$. The continuity of ρ_e allows to apply Lemma 2.8. Similarly for w_e one has

$$\begin{aligned} \|w_e\|_{X_{T_n}} &\leq \left\|\overline{\overline{g}}_w(t)\chi_{[T_{n-1},T_n)} + \overline{\overline{g}}_{w_\varepsilon}\chi_{[0,T_{n-1}]}\right\|_{X_{T_n}} + \|u_{I,\varepsilon}\|_{L^\infty_\omega(\mathbb{R})} \\ &\leq \frac{(\varepsilon\|\partial_t f\|_{L^\infty(0,T_n)} + \overline{p})}{\underline{\mu}} + \|u_{I,\varepsilon}\|_{L^\infty_\omega(\mathbb{R})}. \end{aligned}$$

where χ_A is the characteristic function of the set A, and we used the uniform estimate on $\overline{\overline{g}}_w$ provided by (3.4). These estimates prove that the constant $\overline{c}$ in the contraction in b) is not changing as time evolves. Thus we can fix-point again choosing ΔT_n as in the previous paragraph and prove contraction in $[T_{n-1}, T_n]$. At this step the recursion is complete. The theorem is proven for any positive time.

□

Corollary 3.2.1. *Under the same hypotheses as above, for any pair of positive definite reals* $(\underline{\mu}, \overline{p})$*, the solution-pair* (ϱ_w, w) *solving* (3.1)-(3.2) *satisfies the* a priori *estimates* (1.6).

Proof. We use that

$$\begin{aligned} |\overline{\overline{g}}_w(t)| &\leq \frac{1}{\mu_0(t)}\left\{\varepsilon|\partial_t f| + \min\left(\overline{p}, \left|\min\left(\int_{\mathbb{R}_+}\zeta(w)w\varrho_w da, \overline{p}\right)\right|\right)\right\} \\ &\leq \frac{1}{\mu_0(t)}\left\{\varepsilon|\partial_t f| + \min\left(\overline{p}, \int_{\mathbb{R}_+}\zeta(w)\varrho_w|w|da\right)\right\} \leq \frac{1}{\mu_0(t)}\left\{\varepsilon|\partial_t f| + \int_{\mathbb{R}_+}\zeta(w)\varrho_w|w|da\right\}. \end{aligned}$$

Then the same arguments as in the proof of Lemma 5.1 in [10] provide the *a priori* estimates. Indeed in the sense of characteristics $|w|$ satisfies :

$$\varepsilon\partial_t|w| + \partial_a|w| \leq |\overline{\overline{g}}_w| \leq \frac{1}{\mu_0(t)}\left\{\varepsilon|\partial_t f| + \int_{\mathbb{R}_+}\zeta(w)\varrho_w|w|da\right\}.$$

then multiplying the later inequality by ϱ_w and integrating with respect to age, one gets :

$$\varepsilon\partial_t\int_{\mathbb{R}_+}\varrho_w|w|da + \int_{\mathbb{R}_+}\zeta(w)|w|\varrho_w da \leq \varepsilon|\partial_t f| + \int_{\mathbb{R}_+}\zeta(w)|w|\varrho_w da$$

and because on the right and on the left hand sides the same integral terms cancel, the claim follows. □

Proposition 3.3. *Under Assumptions 2.1, 2.3 and 2.4, let (ϱ_w, w) be the solution of the fully coupled and stabilized problem* (3.1)-(3.2)-(3.3)*, there exists a positive finite constant γ_2 s.t.*

$$\int_{\mathbb{R}_+} \zeta(w(t,a))|w(t,a)|\varrho_w(t,a)da \leq \frac{\gamma_2}{\underline{\mu}}, \quad \forall t \geq 0 \ ,$$

where the constant γ_2 is depends on

- *the* a priori *bound only on $\int_{\mathbb{R}_+} \varrho_w |w| da$ (obtained in Corollary 3.2.1) ,*
- $\|\partial_t f\|_{L^\infty(0,T)}$*,*
- ζ_{Lip}*, and* $\zeta(0)$*.*

Proof. Using equations (3.1), (3.2) and hypotheses 2.1, one has

$$\varepsilon\partial_t(\varrho_w|w|\zeta) + \partial_a(\varrho_w|w|\zeta) + \zeta^2|w|\varrho_w \leq \varrho_w|w|(\varepsilon\partial_t\zeta + \partial_a\zeta) + \zeta\varrho_w|\overline{\overline{g}}_w| \ .$$

Integrating in age and setting $p(t) := \int_{\mathbb{R}_+} \varrho_w(t,a)|w(t,a)|\zeta(w(t,a))\, da$ gives

$$\begin{aligned}
\varepsilon\partial_t p + \int_{\mathbb{R}_+} \zeta^2|w(t,a)|\varrho_w(t,a)da &\leq |\overline{\overline{g}}_w| \left(\zeta_{\text{Lip}} \int_{\mathbb{R}_+} \varrho_w|w|\, da + \int_{\mathbb{R}_+} \zeta(w)\varrho_w(t,a)da \right) \\
&\leq |\overline{\overline{g}}_w| \left(2\zeta_{\text{Lip}} \int_{\mathbb{R}_+} \varrho_w|w|\, da + \zeta(0) \right) \\
&\leq \frac{1}{\underline{\mu}}(\varepsilon|\partial_t f| + p)\left(2\zeta_{\text{Lip}}/\gamma_1 + \zeta(0)\right),
\end{aligned}$$

where $\int_{\mathbb{R}_+} \varrho_w|w|da \leq 1/\gamma_1$. Now we consider the second term in the left hand side above: using Jensen's inequality one writes

$$\left(\frac{\int_{\mathbb{R}_+} \zeta(w)|w(t,a)|\varrho_w(t,a)da}{\int_{\mathbb{R}_+} |w|\varrho_w da} \right)^2 \leq \frac{\int_{\mathbb{R}_+} (\zeta(w))^2|w(t,a)|\varrho_w(t,a)da}{\int_{\mathbb{R}_+} |w|\varrho_w da} \ ,$$

since $|w|\varrho_w / \int_{\mathbb{R}_+} |w|\varrho_w\, da$ is a unit measure. This implies that

$$\int_{\mathbb{R}_+} (\zeta(w))^2|w(t,a)|\varrho_w(t,a)\ da \geq \frac{\left(\int_{\mathbb{R}_+} \zeta(w)|w(t,a)|\varrho_w(t,a)da\right)^2}{\int_{\mathbb{R}_+} |w|\varrho_w da} \geq \gamma_1 p^2 \ .$$

We obtain a Riccati inequality

$$\varepsilon\partial_t p + \gamma_1 p^2 \leq h/\underline{\mu} + p/\underline{\mu} \ , \quad p(0) = \int_{\mathbb{R}_+} \zeta(u_{I,\varepsilon}(a))|u_{I,\varepsilon}(a)|\rho_{I,\varepsilon}(a)\ da \ ,$$

where $h := \varepsilon\|\partial_t f\|_\infty \left(2\zeta_{\text{Lip}}/\gamma_1 + \zeta(0)\right)$ is a constant. We denote by $P_\pm$ the solutions of the steady state equation associated to the last inequality, *i.e.* P solves $\gamma_1 P^2 - P/\underline{\mu} - h/\underline{\mu} = 0$. The solutions are given by

$$P_\pm = \frac{1}{\underline{\mu}}\left(1 \pm \sqrt{1 + 4h\underline{\mu}\gamma_1}\right)/(2\gamma_1) \leq \frac{1}{\underline{\mu}} \max\left(p(0), \left(1 \pm \sqrt{1+4h\gamma_1}\right)/(2\gamma_1)\right) =: \frac{\gamma_2}{\underline{\mu}}.$$

Applying Lemma A.1, we conclude that $p(t) \leq \max\{p(0), P_+\} \leq \gamma_2/\underline{\mu}$, which ends the proof. $\square$

Theorem 3.4. *Suppose that Assumptions 2.1, 2.3 and 2.4 hold, moreover, suppose that $u_{I,\varepsilon} \in L^\infty(\mathbb{R}_+, \omega)$ and that $\|\partial_t f\|_{L^\infty(0,T)}$ is finite, if (ϱ_w, w) is the unique solution of the stabilized problem* (3.1)-(3.2)-(3.3)*, it is also the unique solution of* (3.1)-(3.2) *together with the modified right hand side :*

$$\overline{g}_w = \frac{1}{\max(\mu_{0,w}, \underline{\mu})} \left(\varepsilon\partial_t f + \int_{\mathbb{R}_+} \zeta(w)w\varrho_w\, da \right) \ . \tag{3.5}$$

Proof. The proof is a simple application of the Proposition 3.3 above and taking $\overline{p} > \gamma_2/\underline{\mu}$ when solving (3.1)-(3.2)-(3.3). Indeed, in this case one has that the truncated right hand side from (3.3) becomes (3.5), one since $p(t) := \int_{\mathbb{R}_+} \zeta(w)w\varrho_w da$ never reaches $\pm\overline{p}$. $\square$

4. Impact of the cut-off value on the mean bonds' population

In this section we give ourselves a function $g \in L^\infty(0,T)$ and compute $w \in X_T$ solving (2.3). In what follows we analyze the properties of an age structured model for linkages whose off-rates depend on w : we define ϱ_w as the solution of

$$\begin{cases} \varepsilon \partial_t \varrho_w + \partial_a \varrho_w + \zeta(w)\, \varrho_w = 0\,, & t > 0\,,\ a > 0\,, \\ \varrho_w(t, a=0) = \beta(t)\,(1-\mu_{0,w})\,, & t > 0\,, \\ \varrho_w(t=0,a) = \rho_{I,\varepsilon}(a)\,, & a \geq 0\,, \end{cases} \tag{4.1}$$

where $\mu_{0,w}(t) := \int_{\mathbb{R}_+} \varrho_w(t,\tilde a) d\tilde a$.

We compute a sharper upper bound on $\mu_{0,w}$, namely

Lemma 4.1. *Let Assumptions 2.1 and 2.4 hold. Let ϱ_w be the solution of (4.1). We suppose that $\mu_{0,w}(0) < 1$. Let us fix a positive constant γ_0 s.t.*

$$\gamma_0 < \min\left(1-\mu_{0,w}(0), \frac{\zeta_{\min}}{\zeta_{\min}+\beta_{\max}}\right).$$

Under Assumptions 2.1, 2.3 and 2.4, $\mu_{0,w}(t) < 1-\gamma_0$ holds for every positive time t.

Proof. We proceed similarly as in Lemma 2.2 in [9]. The computations are thus only formal although they can be made rigorous exactly as therein. By hypothesis, the data satisfies $1-\gamma_0-\mu_{0,w}(0) > 0$. By continuity this also holds on a time interval $[0,t_0)$ small enough. We proceed by contradiction and suppose that at time t_0 the mass $\mu_{0,w}(t_0)$ reaches $1-\gamma_0$. The equation on $\mu_{0,w}$ reads:

$$\varepsilon \partial_t \mu_{0,w} - \beta(1-\mu_{0,w}) + \int_{\mathbb{R}_+} \varrho_w(t,a)\zeta(w(t,a))da = 0\,.$$

Multiplying it by -1 and estimating $\beta(t) \leq \beta_{\max}$, one deduces that

$$\varepsilon \partial_t (1-\gamma_0-\mu_{0,w}) + \beta_{\max}(1-\gamma_0-\mu_{0,w}) + \gamma_0\beta_{\max} - \int_{\mathbb{R}_+} \varrho_w(t,a)\zeta(w(t,a))da \geq 0\,,$$

then the lower bound on ζ implies

$$\varepsilon \partial_t (1-\gamma_0-\mu_{0,w}) + \beta_{\max}(1-\gamma_0-\mu_{0,w}) + \gamma_0\beta_{\max} \geq \zeta_{\min}\mu_{0,w}\,.$$

We transform the latter right hand side writing

$$\zeta_{\min}\mu_{0,w} = -\zeta_{\min}(1-\gamma_0-\mu_{0,w}) + \zeta_{\min}(1-\gamma_0)\,.$$

Setting $q(t) := (1-\gamma_0-\mu_{0,w}(t))$, one then has

$$\varepsilon \partial_t q + (\zeta_{\min}+\beta_{\max})q \geq \zeta_{\min} - (\zeta_{\min}+\beta_{\max})\gamma_0 > 0\,,$$

the latter estimate being true under the hypothesis that $\gamma_0 < \zeta_{\min}/(\zeta_{\min}+\beta_{\max})$. The conclusion then follows integrating the latter inequality in time

$$q(t_0) > \exp\left(-\frac{(\beta_{\max}+\zeta_{\min})t_0}{\varepsilon}\right) q(0) > 0,$$

under the hypothesis that $\gamma_0 < (1-\mu_{0,w}(0))$. But this contradicts the assumption that $q(t_0) = 0$, which ends the proof. ☐

We do not have a positive definite lower bound on $\mu_{0,w}$ yet : at this stage we only know that $\mu_{0,w}(t) \geq 0$. For this reason we define $\tilde\varrho^\delta_w(t,a) := \varrho(t,a)/(\mu_{0,w}(t)+\delta)$ and we observe that this new function is in

$L^\infty_{\rm loc}((0,T)\times\mathbb{R}_+)$. It solves the equation

$$
\begin{cases}
\varepsilon\partial_t\tilde{\varrho}^\delta_w+\partial_a\tilde{\varrho}^\delta_w+\left(\zeta-\int_{\mathbb{R}_+}\zeta\tilde{\varrho}^\delta_w\right)\tilde{\varrho}^\delta_w \\
\qquad\qquad +\beta\left(\frac{1}{\mu_{0,w}+\delta}-\frac{\mu_{0,w}}{\mu_{0,w}+\delta}\right)\tilde{\varrho}^\delta_w=0\,, \quad t>0\,,\ a>0\,, \\
\tilde{\varrho}^\delta_w(t,a=0)=\beta(t)\left(\frac{1}{\mu_{0,w}+\delta}-\frac{\mu_{0,w}}{\mu_{0,w}+\delta}\right)\,, \qquad t>0\,, \\
\tilde{\varrho}^\delta_w(t=0,a)=\rho_{I,\varepsilon}(a)/(\mu_{0,w}+\delta)\,, \qquad a\geq 0\,.
\end{cases}
\tag{4.2}
$$

When $\delta=0$ one denotes $\tilde{\varrho}^\delta_w$ simply by $\tilde{\varrho}_w$.

Proposition 4.2. *Let $g\in L^\infty(0,T)$ be given, and let (ϱ,w) be the solutions of (4.1)-(2.3). Under Assumptions 2.1, 2.3 and 2.4 and if $\mu_{0,w}(0)\leq 1-\gamma_0$, there exists a constant $\bar{\zeta}$ independent of δ and ε such that for every positive δ it holds that*

$$
\int_{\mathbb{R}_+}\zeta(w(t,a))\,\tilde{\varrho}^\delta_w(t,a)da\leq\bar{\zeta}+\zeta_{\rm Lip}\,\|g\|_{L^\infty(0,T)}\,\min\left(\frac{2}{\gamma_0\beta_{\min}},\frac{T}{\varepsilon}\right)\,,\quad \forall t\geq 0,
$$

where $\bar{\zeta}:=\zeta(0)+\int_{\mathbb{R}_+}\zeta(u_{I,\varepsilon}(a))\tilde{\rho}_{\varepsilon,I}(a)da$. Taking the limit as δ goes to 0, one obtains then the analogous result for $\tilde{\varrho}_w$.

Proof. The product $p(t,a):=\zeta\tilde{\varrho}^\delta_w$ satisfies

$$
\varepsilon\partial_t p+\partial_a p+\left(\zeta^2-\zeta\int_{\mathbb{R}_+}\zeta\tilde{\varrho}^\delta_w\right)\tilde{\varrho}^\delta_w+\tilde{\varrho}^\delta_w(t,0)p=\zeta'(w)g(t)\tilde{\varrho}^\delta_w\,.
$$

Indeed, using arguments as in Lemma 2.1 p. 489 and Lemma 3.1 p. 493 [9], one proves that if w solves (2.3) and ζ is Lipschitz, then $\zeta(w)$ solves $(\varepsilon\partial_t+\partial_a)\zeta(w)=\zeta'(w)g$ in the sense of characteristics (as in Theorem 2.5) with the corresponding boundary conditions. Then the latter equation on p is understood in the same manner.

Integrating in age and setting $q(t):=\int_{\mathbb{R}_+}p(t,a)da$ we conclude that

$$
\varepsilon\partial_t q-\zeta(t,0)\tilde{\varrho}^\delta_w(t,0)+\int_{\mathbb{R}_+}\zeta^2\tilde{\varrho}^\delta_w da-\left(\int_{\mathbb{R}_+}\zeta\tilde{\varrho}^\delta_w\right)^2+q\tilde{\varrho}^\delta_w(t,0)\leq\zeta_{\rm Lip}\|g\|_\infty\,.
\tag{4.3}
$$

To find a lower bound for $\tilde{\varrho}^\delta_w(0,t)$ we choose $\delta<\gamma_0/2$ and use the upper bound on $\mu_{0,w}(t)$ established in Lemma 4.1 in order to obtain

$$
\tilde{\varrho}^\delta_w(0,t)\geq\beta_{\min}\left(\frac{1}{1-\gamma_0+\delta}-1\right)\geq\beta_{\min}\frac{\gamma_0}{2}\,.
\tag{4.4}
$$

Assuming $\mu_{0,w}(t)>0$ we also find using Jensen's inequality that

$$
\left(\int_{\mathbb{R}_+}\zeta(w(t,a))\tilde{\varrho}^\delta_w(t,a)da\right)^2\leq\int_{\mathbb{R}_+}(\zeta(w(t,a)))^2\,\tilde{\varrho}^\delta_w da\frac{\mu_{0,w}}{(\mu_{0,w}+\delta)}\leq\int_{\mathbb{R}_+}(\zeta(w(t,a)))^2\,\tilde{\varrho}^\delta_w da\,.
$$

If $\mu_{0,w}(t)=0$ the same inequality holds true since then $\varrho(t,a)=0$ for almost every a. These considerations allow then to rewrite (4.3) as

$$
\varepsilon\partial_t q+\tilde{\varrho}^\delta_w(0,t)(q-\zeta(0))\leq\zeta_{\rm Lip}\|g\|_\infty\,.
$$

Setting $\tilde{q}:=q-\zeta(0)$ and using Gronwall's Lemma gives

$$
\tilde{q}(t)\leq\exp\left(-\frac{1}{\varepsilon}\int_0^t\tilde{\varrho}^\delta_w(0,s)ds\right)\tilde{q}(0)+\frac{\zeta_{\rm Lip}\|g\|_\infty}{\varepsilon}\int_0^t\exp\left(-\frac{1}{\varepsilon}\int_\tau^t\tilde{\varrho}^\delta_w(0,s)ds\right)d\tau\,.
$$

Thanks to the uniform lower bound (4.4) we conclude

$$
\tilde{q}(t)\leq\exp\left(-\frac{\beta_{\min}\gamma_0 t}{2\varepsilon}\right)\tilde{q}(0)+\frac{2\zeta_{\rm Lip}\|g\|_\infty}{\gamma_0\beta_{\min}}\left(1-\exp\left(-\frac{\beta_{\min}\gamma_0 t}{2\varepsilon}\right)\right)\,,
$$

which then gives turning to the variable q that

$$q(t) \leq \zeta(0) + \int_{\mathbb{R}_+} \zeta(u_{I,\varepsilon}(a))\tilde{\rho}_{\varepsilon,I}(a)da + \frac{2\zeta_{\mathrm{Lip}}\|g\|_\infty}{\gamma_0\beta_{\min}}\left(1-\exp\left(-\frac{\beta_{\min}\gamma_0 t}{2\varepsilon}\right)\right) . \tag{4.5}$$

This bound is uniform in δ. One passes to the limit $\delta = 0$ which gives the final result. $\square$

Proposition 4.3. *Under the same assumptions as above and if $\mu_{0,w}(t) < 1-\gamma_0$ and choosing $\mu_{0,\min}$ s.t.*

$$\mu_{0,\min} < \min\left(\mu_{0,w}(0), \frac{\beta_{\min}}{\beta_{\min} + \bar{\zeta} + \zeta_{\mathrm{Lip}}\|g\|_{L^\infty(0,T)}\min\left(\frac{2}{\gamma_0\beta_{\min}}, \frac{T}{\varepsilon}\right)}\right) ,$$

where we used the bound provided by Proposition 4.2, one has a lower bound on $\mu_{0,w}$:

$$\mu_{0,w}(t) \geq \mu_{0,\min} , \quad \forall t \geq 0 .$$

Proof. We integrate the equation (1.1) with respect to age which gives

$$\begin{cases} \varepsilon\partial_t\mu_{0,w} - \beta(1-\mu_{0,w}) + \int_{\mathbb{R}_+} \varrho_w(t,a)\zeta(w(t,a))da = 0 , & t > 0 , \\ \mu_{0,w}(0) = \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a)da , & t = 0 . \end{cases}$$

we divide and we multiply the last term on the left hand side by $\mu_{0,w}$ and we write:

$$\varepsilon\partial_t\mu_{0,w} + \left(\beta(t) + \int_{\mathbb{R}_+}\zeta\tilde{\varrho}_w da\right)\mu_{0,w} = \beta(t) , \quad t > 0 .$$

Now suppose that there exists a time small enough s.t. $\mu_{0,w}(t) > \mu_{0,\min}$ for all $t \in [0,t_0)$ and that $\mu_{0,w}(t_0) = \mu_{0,\min}$. We use the notation $\overline{\lambda} := \bar{\zeta} + \zeta_{\mathrm{Lip}}\|g\|_{L^\infty(0,T)}\min\left(\frac{2}{\gamma_0\beta_{\min}}, \frac{T}{\varepsilon}\right)$ and write for the difference $\tilde{\mu}_0(t) := \mu_{0,w}(t) - \mu_{0,\min}$

$$\begin{aligned} \varepsilon\partial_t\tilde{\mu}_0 + \left(\beta(t) + \int_{\mathbb{R}_+}\zeta\tilde{\varrho}_w da\right)\tilde{\mu}_0 &= \beta(t)(1-\mu_{0,\min}) - \left(\int_{\mathbb{R}_+}\zeta\tilde{\varrho}_w da\right)\mu_{0,\min} \geq \\ &\geq \beta_{\min}(1-\mu_{0,\min}) - \overline{\lambda}\mu_{0,\min} = \left(\beta_{\min} + \overline{\lambda}\right)\left(\frac{\beta_{\min}}{\beta_{\min}+\overline{\lambda}} - \mu_{0,\min}\right) > 0 , \end{aligned}$$

which holds thanks to the bound on $\int_{\mathbb{R}_+}\zeta\tilde{\varrho}_w\, da$ established in Proposition 4.2 and the definition of $\mu_{0,\min}$. Using Gronwall's Lemma we finally obtain that

$$\mu_{0,w}(t_0) - \mu_{0,\min} > \exp\left(-\frac{1}{\varepsilon}\int_0^{t_0}(\beta(\tau)+\overline{\lambda})d\tau\right)(\mu_{0,w}(0) - \mu_{0,\min}) > 0 ,$$

which contradicts the fact that $\mu_{0,w}(t_0) = \mu_{0,\min}$. This ends the proof. $\square$

5. Local existence of the fully coupled problem

Theorem 5.1. *Let f be a Lipschitz function on $(0,T)$ and $u_{I,\varepsilon} \in L^\infty(\mathbb{R}_+,\omega)$. We suppose that Assumptions 2.1, 2.3 and 2.4 hold. Let (ϱ_w, w) be the solution of* (3.1)-(3.2) *together with $\overline{g}_w$, the simple cut-off defined by* (3.5). *Then for any fixed $\underline{\mu} < \mu_{0,w}(0)$ there exists a time*

$$T = \frac{\varepsilon}{\gamma_3}\left(\beta_{\min}\underline{\mu} - (\beta_{\min}+\bar{\zeta})\underline{\mu}^2\right)$$

for which $\mu_{0,w}(t) > \underline{\mu}$ for any $t \in (0,T)$. So the solution (ϱ, w) of (3.1)-(3.2)-(3.5) *is also the unique local solution of the fully coupled system* (1.1)-(1.3).

Proof. Gathering results above one has :

$$\|\overline{g}_w\|_{L^\infty(0,T)} \leq \frac{1}{\underline{\mu}}(\varepsilon|\partial_t f| + p(t)) \leq \frac{1}{\underline{\mu}}\left(\varepsilon\|\partial_t f\|_{L^\infty(0,T)} + \frac{\gamma_2}{\underline{\mu}}\right) \leq \frac{\gamma_3}{\underline{\mu}^2},$$

since we suppose that $\underline{\mu} < 1$ and we set $\gamma_3 := \zeta_{\mathrm{Lip}}(\varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \gamma_2)$. Thanks to Proposition 4.3, the lower bound on $\mu_{0,w}$ then becomes :

$$\mu_{0,w}(t) > \min\left(\mu_{0,w}(0), \frac{\beta_{\min}\underline{\mu}^2}{(\beta_{\min} + \bar{\zeta})\underline{\mu}^2 + \frac{\gamma_3 T}{\varepsilon}}\right).$$

Choosing $\underline{\mu} < \mu_{0,w}(0)$ we tune T s.t.

$$\frac{\beta_{\min}\underline{\mu}^2}{(\beta_{\min} + \bar{\zeta})\underline{\mu}^2 + \frac{\gamma_3 T}{\varepsilon}} > \underline{\mu}$$

□

6. Global existence for specific data

Under hypotheses of Theorem 3.2, whatever be the time of existence T for (ϱ_w, w), the solutions of the stabilized model, then thanks to Corollary 3.2.1 one has that :

$$\begin{aligned}\int_{\mathbb{R}_+} \zeta(w(t,a))\varrho(t,a)da &\leq \int_{\mathbb{R}_+} (\zeta(0) + \zeta_{\mathrm{Lip}}|w|)\varrho_w da \\ &\leq \zeta(0) + \zeta_{\mathrm{Lip}}\left(\int_{\mathbb{R}_+} |u_{I,\varepsilon}|\rho_{I,\varepsilon} da + \int_0^T |\partial_t f| ds\right) =: \breve{\zeta}, \quad \forall t \in (0,T).\end{aligned}$$

Proposition 6.1. *Under assumptions 2.1, 2.3 and 2.4, if $\beta_{\min} > \breve{\zeta}$ and we set :*

$$0 < \mu_{0,\min} < \min\left(1 - \frac{\breve{\zeta}}{\beta_{\min}}, \mu_{0,w}(0)\right)$$

then one has

$$\mu_{0,w}(t) \geq \mu_{0,\min}, \quad \forall t \in (0,T).$$

Proof. We set $\hat{\mu} := \mu_{0,w}(t) - \mu_{0,\min}$ and write the equation that it satisfies :

$$\varepsilon\partial_t\hat{\mu} + \beta\hat{\mu} = -\int_{\mathbb{R}_+} \zeta\varrho da + \beta(1 - \mu_{0,\min}) \geq -\breve{\zeta} + \beta_{\min}(1 - \mu_{0,\min}).$$

We estimate from below the right hand side using previous bounds. The lower bound is positive definite provided that $\beta_{\min} > \breve{\zeta}$ and that $\mu_{0,\min} < 1 - \breve{\zeta}/\beta_{\min}$. Using Gronwall's Lemma, one has :

$$\hat{\mu}(t) \geq \exp(-\beta_{\max} t/\varepsilon)\hat{\mu}(0) > 0$$

if $\mu_{0,\min} < \mu_{0,w}(0)$, which ends the proof. □

Theorem 6.2. *If we fix a finite time $T > 0$. Under Assumptions 2.1 and 2.3, and assuming that*

i) f is Lipschitz on $(0,T)$,

ii) β satisfies assumptions 2.4 together with $\beta_{\min} > \breve{\zeta}$

there exists a unique solution $(\rho_\varepsilon, u_\varepsilon) \in C(0,T; L^1(\mathbb{R}_+)) \times X_T$ solving system (1.1)-(1.3).

Proof. By Theorem 3.4, there exists a unique couple $(\varrho_w, w) \in C(0,\infty; L^1(\mathbb{R}_+)) \times X_\infty$ solving (3.1)-(3.2)-(3.5) for any given constant $\underline{\mu}$. We choose $T > 0$ and provided that β satisfies hypothesis required by Proposition 6.1 we set the constants $0 < \underline{\mu} < \mu_{0,\min}$ according to Propositions 6.1. Then $\mu_{0,w}$ does not reach the threshold value $\underline{\mu}$ so that

$$\begin{aligned}\overline{g}_w(t) =& \frac{1}{\max(\mu_{0,w}(t), \underline{\mu})}\left(\varepsilon\partial_t f + \int_{\mathbb{R}_+} (\zeta(w)\varrho_w w)(t,a)da\right) = \\ & \frac{1}{\mu_{0,w}(t)}\left(\varepsilon\partial_t f + \int_{\mathbb{R}_+} (\zeta(w)\varrho_w w)(t,a)da\right) = g_w(t), \quad a.e.\ t \in (0,T).\end{aligned}$$

The pair (ϱ_w, w) is in fact also solving (1.1)-(1.3) on this time interval. This provides existence of a solution $(\rho_\varepsilon, u_\varepsilon) = (\varrho_w, w)$ on $[0,T]$. Since by Theorem 3.4 (ϱ_w, w) is unique, so is $(\rho_\varepsilon, u_\varepsilon)$ in this time period. □

7. Blow up for positive solutions

Theorem 7.1. *Under assumptions 2.3 and if T_0 is the time of existence of $(\rho_\varepsilon, u_\varepsilon)$ solving* (1.1)-(1.3)*, and if*

i) $u_{I,\varepsilon}(a) \geq 0$ *for a.e.* $a \in \mathbb{R}_+$,
ii) $\partial_t f(t) > 0$ *for a.e.* $t \in (0, T_0)$,

then the product $\rho_\varepsilon(t,a)u_\varepsilon(t,a)$ is non-negative for a.e. $(t,a) \in (0,T_0) \times \mathbb{R}_+$.

Proof. Since it holds that $f(0) = \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a) u_{I,\varepsilon}(a)\, da$ and $f(t) = \int_{\mathbb{R}_+} \rho_\varepsilon(t,a)u_\varepsilon(t,a)\, da$ yields

$$\int_{\mathbb{R}_+} \rho_\varepsilon(t,a)|u_\varepsilon(t,a)| da \leq \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a)|u_{I,\varepsilon}(a)| da + \int_0^t |\partial_t f(\tilde t)| d\tilde t =$$

$$= \int_{\mathbb{R}_+} \rho_{I,\varepsilon}(a) u_{I,\varepsilon}(a) da + \int_0^t \partial_t f(\tilde t)\, d\tilde t = f(t) = \int_{\mathbb{R}_+} \rho_\varepsilon(t,a) u_\varepsilon(t,a)\, da\,,$$

which implies the result. □

Lemma 7.2. *Suppose that ζ_c is a convex locally differentiable function. Then for any function $u \in X_\infty$, one has :*

$$\zeta_c'(0) \int_{\mathbb{R}_+} u \rho_\varepsilon da \leq \int_{\mathbb{R}_+} \zeta_c(u(t,a)) \rho_\varepsilon(t,a) da - \zeta_c(0)\mu_0, \qquad a.e.\ t \in \mathbb{R}_+$$

Proof. Since ζ_c is convex, for almost every (t,a), one has :

$$\zeta_c'(0)(u(t,a) - 0) \leq \zeta_c(u(t,a)) - \zeta(0)$$

and integrating with respect to $\rho_\varepsilon da$, one has the desired result. □

Proposition 7.3. *Under assumptions 2.3 and 2.4 and if*

i) ζ *satisfies Assumptions 2.1 and admits a lower convex envelop* ζ_c *s.t.* $\zeta_c(u) \leq \zeta(u)$ *for all* $u \in \mathbb{R}_+$ *with* $\zeta_c'(0) > 0$,
ii) let f *be a Lipschitz function s.t.* $\partial_t f(t) > 0$ *for a.e.* $t \in (0, T)$,
iii) f *and* β *are s.t.* $\beta_{\max} < \zeta_c'(0) f_{\min}$,
iv) $u_{I,\varepsilon}(a) \geq 0$ *for a.e.* $a \in \mathbb{R}_+$,

then if the solution $(\rho_\varepsilon, u_\varepsilon)$ solving (1.1)-(1.3) *exists until a finite time T_0, this time cannot be greater than*

$$t_0 := \frac{\varepsilon}{\beta_{\min} + \zeta_c(0)} \ln \left(1 + \frac{\mu_0(0)(\beta_{\min} + \zeta_c(0))}{\zeta_c'(0) f_{\min} - \beta_{\max}} \right)$$

for which

$$\mu_0(t_0) \leq 0.$$

Moreover, on $(0, t_0) \times \mathbb{R}_+$, one has a lower bound on the profile of u_ε namely

$$u_\varepsilon(t,a) \geq \varepsilon \gamma_6 \ln \left(1 + \frac{\min(t, \varepsilon a)}{(t_0 - t)} \right),$$

where $\gamma_6 := t_0 \inf_{t \in (0,t_0)} \partial_t f / \mu_0(0)$.

Proof. By Theorem 7.1 $u_\varepsilon(t,a) \geq 0$ a.e. $(t,a) \in (0,T_0) \times \mathbb{R}_+$. The equation for μ_0 reads :

$$\varepsilon \partial_t \mu_0 - \beta(1 - \mu_0) + \int_{\mathbb{R}_+} \zeta(u_\varepsilon(t,a)) \rho_\varepsilon(t,a) da = 0$$

that we estimate using Lemma 7.2 as follows :

$$\varepsilon \partial_t \mu_0 - \beta(1 - \mu_0) + \zeta_c'(0) \int_{\mathbb{R}_+} u_\varepsilon(t,a) \rho_\varepsilon(t,a) da + \zeta_c(0)\mu_0 \leq 0$$

and becomes under these simplifications :

$$\varepsilon \partial_t \mu_0 - \beta(1 - \mu_0) + \zeta_c(0)\mu_0 + \zeta_c'(0) f \leq 0. \tag{7.1}$$

We can deduce from this equation that

$$\varepsilon \partial_t \mu_0 + (\beta_{\min} + \zeta_c(0))\mu_0 \leq \beta_{\max} - \zeta_c'(0) f_{\min}$$

which gives using Gronwall's Lemma that $\mu_0(t) \leq \overline{\overline{\mu}}(t)$, where

$$\overline{\overline{\mu}}(t) := \mu_0(0)\exp\left(-\frac{(\beta_{\min}+\zeta_c(0))}{\varepsilon}t\right) - \frac{\zeta_c'(0)f_{\min}-\beta_{\max}}{(\beta_{\min}+\zeta_c(0))}\left(1-\exp\left(-\frac{(\beta_{\min}+\zeta_c(0))}{\varepsilon}t\right)\right).$$

Looking for the time t_0 s.t. $\overline{\overline{\mu}}(t_0)=0$ provides the explicit form of t_0 in the claim. Thus $T_0 < t_0$. Moreover, as $\overline{\overline{\mu}}(t)$ is a convex function one has that :

$$\mu_0(t) \leq \left(1-\frac{t}{t_0}\right)\overline{\overline{\mu}}(0) + \frac{t}{t_0}\overline{\overline{\mu}}(t_0) \equiv \left(1-\frac{t}{t_0}\right)\overline{\overline{\mu}}(0),$$

and because, by Lemma 7.2, $\zeta(u_\varepsilon)u_\varepsilon\rho_\varepsilon$ is positive almost everywhere on $(0,t_0)\times\mathbb{R}_+$,

$$\varepsilon\partial_t u_\varepsilon + \partial_a u_\varepsilon \geq \frac{\varepsilon\partial_t f}{\mu_0(t)} \geq \frac{\varepsilon\gamma_6}{t_0-t}, \qquad \text{a.e in } (0,t_0)\times\mathbb{R}_+.$$

Using Duhamel's formula provides

$$u_\varepsilon(t,a) \geq \begin{cases} \varepsilon\gamma_6\int_{-a}^0 \frac{ds}{t_0-(t+\varepsilon s)}ds, & \text{if } t\geq\varepsilon a,\\ u_{I,\varepsilon}(a-t/\varepsilon)+\varepsilon\gamma_6\int_{-t/\varepsilon}^0\frac{ds}{t_0-(t+\varepsilon s)}ds & \text{otherwize,}\end{cases}$$

which then gives the lower estimate on u_ε. □

Appendix A. Riccati inequalities

Lemma A.1. *Let $\varepsilon > 0$ and real, let y be a positive differentiable function of $t\in\mathbb{R}_+$, satisfying*

$$\begin{cases} \varepsilon\partial_t y + Ay^2 \leq By + C, & t>0,\\ y(0)=y_0, & t=0\end{cases}$$

where $y_0>0$ and $(A,B,C)\in(\mathbb{R}_+)^3$. Setting $y_+ := (B+\sqrt{B^2+4AC})/(2A)$, one has that

$$y(t) \leq \max(y_0,y_+), \quad \forall t\in\mathbb{R}_+.$$

Proof. We set $m := \max(y_0,y_+)$, it satisfies $-Am^2+Bm+C\leq 0$. Then we define $\tilde{y} := y-m$ which then solve the differential inequality :

$$\varepsilon\partial_t\tilde{y} + A\tilde{y}^2 + (2mA-B)\tilde{y} \leq 0, \tag{A.1}$$

Since the quadratic term is positive we neglect it, and apply Gronwall's Lemma :

$$\tilde{y}(t) \leq \exp\left(-\frac{(2Am-B)t}{\varepsilon}\right)\tilde{y}(0) = \exp\left(-\frac{(2Am-B)t}{\varepsilon}\right)(y_0-m) \leq 0$$

which ends the proof. □

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